**Covariance Based Uncertainty Analysis with Unscented Transformation**

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**Abstract** — Application of the unscented transformation (UT) and higher order unscented transformation (HOUT) are considered for uncertainty analysis. Using the principle that a set of discretely sampled points can be used to calculate mean and covariance, we can analyze nonlinear systems without the linearization steps and without defining the Jacobian matrix. An important example is presented.

**Index Terms** — Measurement uncertainty, standard deviation, unscented transformation.

I. INTRODUCTION

Reference [1] presents a new uncertainty analysis approach for vector network analyzer (VNA) scattering-parameter measurements based on the covariance matrix description. We propose a new method for covariance matrix calculations for this application.

The problem of propagating an uncertainty can be expressed in the following form. It is assumed that $x$ is a random variable with mean $m_x$ and covariance matrix $V_x$. A random variable $y$ is related to $x$ through the function $y = f(x)$. (1)

This model is, in general, nonlinear. The mean $m_y$ and covariance matrix $V_y$ of $y$ are to be calculated. Normally, uncertainty analysis uses only the covariance matrix. In this paper we consider several ways to calculate both statistical parameters and perform comparisons with specific examples.

II. THE UNSCENTED TRANSFORMATION

References [2]–[6] describe the principle of the unscented transformation (UT). The probability density function of an $n$-dimensional variable $x$ is approximated by $(2n+1)$ sigma points $X_i (i = 0...2n)$. The statistics of the sigma points set and their weights are selected to match the statistics of the input $x$.

Coordinates of the sigma points in $n$-dimensional space are defined as follows:

\[ X_0 = m_x, \]
\[ X_i = m_x + \gamma \cdot (\sqrt{V_x})_i, \quad \text{for } i = 1,..,n, \]
\[ X_i = m_x - \gamma \cdot (\sqrt{V_x})_i, \quad \text{for } i = n+1,..,2n, \]
\[ \gamma = \sqrt{n + \lambda}, \]

where $\lambda$ is a scaling parameter such that $\lambda = \alpha^2 \cdot (n + k) - n$, and $\alpha$ is a positive coefficient less than 1. The constant $k$ can be chosen with the condition of $k = (3 - n)$. Expressions (3) and (4) require the square root of the matrix $V_x$ (index – the column number). The root can be obtained by Cholesky factorization.

The transformation procedure consists of two steps. First, each point is propagated through the function to yield a set of transformed sigma points $Y_i = f[X_i]$. Second, the mean and covariance matrix are found from a weighted average of the transformed points:

\[ m_y = \frac{\sum_{i=0}^{2n} W_i^{(m)} Y_i}{\sum_{i=0}^{2n} W_i^{(m)}}, \]
\[ V_y = \frac{\sum_{i=0}^{2n} W_i^{(c)} (Y_i - m_y) (Y_i - m_y)^T}{\sum_{i=0}^{2n} W_i^{(c)}}, \]

$W_i$ denotes the weight associated with the $i$th point. A variety of coefficients have been provided to calculate the mean and covariance matrix using indexes $(m)$ and $(c)$. The weights are given by:

\[ W_0^{(m)} = \lambda f(n + \lambda), \]
\[ W_0^{(c)} = \lambda f(n + \lambda) + (1 - \alpha^2 + \beta), \]
\[ W_i^{(m)} = W_i^{(c)} = f\left[ 2 \cdot (n + \lambda) \right], \quad i = 1,..,2n. \]

For a Gaussian distribution, the optimal value of $\beta = 2 [5,7]$. The sum of weights to calculate the mean is 1.

Equations (3)-(10) define a scaled unscented transformation [5]–[7]. For $\alpha = 1$ and $\beta = 0$, we obtain the standard unscented transformation [2]–[4].

Reference [8] presents the principle of the higher order unscented transformation (HOUT). The HOUT retains the same structure as the original UT. The HOUT consists of an augmented sigma points set with separate weights.

Unscented transformation uses a set of $(2n+1)$ sigma points. To perform HOUT we need $(4n+1)$ sigma points. The values
of the weighting coefficients are chosen such that the initial set of sigma points and these coefficients allow for accurate calculation of the moments of the initial Gaussian variable. As a simple example, the paper [8] presents the results of the scalar problem solutions. A diagonal matrix $V_x$ of these results can be easily extended to the multidimensional case.

III. EXAMPLE 1: CARTESIAN TO POLAR COORDINATE TRANSFORMATION

In order to verify that the unscented transformation (6), (7) holds, we verify it and its linearization with Monte-Carlo simulation. In this section we consider the simple case of converting complex scattering parameters from the real-imaginary (Re-Im) plane to amplitude-phase (A-Ph) plane:

$$
\mathbf{x} = \begin{bmatrix} \text{Re} \\ \text{Im} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} A \\ \text{Ph} \end{bmatrix} = \begin{bmatrix} \sqrt{\text{Re}^2 + \text{Im}^2} \\ \text{atan}(\text{Im}/\text{Re}) \end{bmatrix}.
$$

The mean and covariance matrix of the vector $\mathbf{x}$ are defined as:

$$
\mathbf{m}_x = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}, \quad V_x = \begin{bmatrix} 0.005^2 & 0 \\ 0 & 0.005^2 \end{bmatrix}.
$$

Calculation was performed of the statistical characteristics (mean and covariance matrix) of the vector $\mathbf{y}$ in four ways: by the Monte-Carlo simulation (MCS) when averaged over a sample of $N = 5 \cdot 10^3$ points; by the linearization method (LIN) [1]; and using the unscented transformation (UT) [4] and the higher order unscented transformation (HOUT) [8].

The resulting uncertainties are illustrated and analyzed below. Fig. 1-3 show the results of comparing the calculated values of the moments (mean and covariance ellipse).

![Fig. 1. Comparison of means and covariances (3σ): the solid line is from Monte-Carlo simulation; the dotted line is from linearisation.](image1)

![Fig. 2. Comparison of means and covariances (3σ): the solid line is from Monte-Carlo simulation; the dotted line is from unscented transformation.](image2)

![Fig. 3. Comparison of means and covariances (3σ): the solid line is from Monte-Carlo simulation; the dotted line is from higher order unscented transformation.](image3)

The true values of $\mathbf{m}_y$ and $V_y$ are obtained by performing Monte Carlo simulation. Fig. 1 shows a comparison of the statistical characteristics when linearization is performed. Fig. 2 shows a comparison of the statistical characteristics found using unscented transformation. Fig. 3 shows a comparison of the statistical characteristics when a higher order unscented transformation is performed.
Comparative analysis of the results shows that in this example, use of linearization leads to errors in calculating the mean and covariance matrix. The mean and covariance matrix obtained by UT and HOUT are closer to the true value. For a given covariance matrix of $x$, the linearization error will depend on the average level of the desired signal $S = |m_x|$.

Fig. 4 illustrates the mean of amplitude depending on $S$ (at various signal to noise ratios) using various methods. Fig. 5 shows the standard deviation (SD) of the dependence of the amplitude on $S$ for different methods. If only the linear term of the Taylor series is considered, for small values of $S$, the amount of error in determination of the statistical properties of the amplitude is increased.

![Fig. 4](image)

**Fig. 4.** The means of the amplitude obtained by different methods.

![Fig. 5](image)

**Fig. 5.** The standard uncertainties of the amplitude obtained by different methods.

The results shown in Fig. 4 and 5 were obtained under the following conditions: Re=Im for all $S$ and diagonal covariance matrix with the value of $0.01^2$. Unscaled UT was applied. Monte Carlo estimates were obtained for $N = 5 \times 10^6$ points.

IV. EXAMPLE 2: LOSS COMPARISON METHOD WITH REFLECTIONLESS STANDARD POWER METER AND SIGNAL GENERATOR IN POWER METER CALIBRATION

Another example of measurements containing a nonlinear transformation is calibration of a microwave power meter with a mismatch at its input. The model of the measured signal is $M = 1 - |\Gamma|^2$, 

where $\Gamma$ is the reflection coefficient of the power meter under test. The model (13) is obtained for the specific case of the loss comparison method for power meter calibration. The standard power meter and the signal generator are assumed to be reflectionless [9]. Consider the transformation (13) in detail.

The input quantities are the real (Re) and imaginary (Im) components of $\Gamma$, the output quantity is the mismatch factor $M$:

$$
x = \begin{bmatrix} \text{Re} \\ \text{Im} \end{bmatrix}, \quad y = M = 1 - \text{Re}^2 - \text{Im}^2.
$$

The mean $m_x$ and covariance $V_x$ of the vector $x$ are determined by the reflection coefficient uncertainty.

An unscented transformation (including higher order UT), linearization and Monte-Carlo simulation are applied to this example in order to determine whether the accuracy of uncertainty analysis based on unscented transformation is superior to linearization.

Monte Carlo simulation (MCS) is used to generate a probability density function for $y$. Increasing the number of tests $N$ leads to better estimates of mean and covariance of $y$. MCS estimates are efficient and unbiased. This means that when $N$ is large, they tend to true values of mean $m_y$ and covariance $V_y$. However, MCS requires significant computational resources. UT in different variants is less demanding on the time of calculation. Linearization allows for very fast calculation, but this method requires an analytical expression of the Jacobian matrix. In this example, the Jacobian takes the simple form:

$$
J = \begin{bmatrix} -2 \text{Re} & -2 \text{Im} \end{bmatrix}.
$$

Calculation of the dispersion of $y$ is performed by means of the Jacobian at the average $x$:

$$
V_y = J \cdot V_x \cdot J^T = 4 \cdot \left[ \sigma_{\text{re}}^2 m_{\text{re}}^2 + \sigma_{\text{im}}^2 m_{\text{im}}^2 + 2 \rho \sigma_{\text{re}} \sigma_{\text{im}} m_{\text{re}} m_{\text{im}} \right],
$$

where $m_x$ and $\sigma_x$ are the mean and root-mean square values of the corresponding components of the input vector and $\rho$ represents the correlation coefficient of the components. Fig. 6 shows the SD of $y$ as a function of $S = |m_x|$. Fig. 7 shows the relative error of the SD estimate, determined with the formula:

$$
E_{\sigma} = 20 \cdot \log_{10} \left( \frac{\sigma_x - \sigma_{\text{MCS}}}{\sigma_{\text{MCS}}} \right),
$$

where $\sigma_x$ is the estimate of SD for a given method and $\sigma_{\text{MCS}}$ is the estimate of SD obtained by MCS.
V. CONCLUSION

This paper has considered one specific application of unscented transformations. The presented methods allow for taking account of higher order terms in the Taylor series expansion, and calculating the full covariance matrix (including correlation between variables). The UT is better suited than the linearization for the vector-network-analyzer uncertainty analysis at low signal levels. The UT and HOUT can predict the mean and covariance matrix with second or higher order accuracy, but do not require derivation of Jacobians or Hessians. For high signal level (or low noise), or linear models, the unscented transformations give the same results as linearization.

The performance benefits of the unscented transformation were demonstrated in a realistic example.

Relative to the other methods, software using the unscented transformation is much simpler to develop.

REFERENCES


